4.3 Termination with Reduction Relations

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Goal: develop techniques to prove termination of arbitrary TRSs which succeed as often as possible.
automatically

General idea: Find a well-founded relation $\succ$ such that $\succ$ contains the rewrite relation (i.e., if $s \to_{\mathcal{R}} t$ then $s \succ t$).

This proves termination of $\mathcal{R}$:

$t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \dots$ would imply

$t_0 \succ t_1 \succ t_2 \succ \dots$ which contradicts well-foundedness of $\succ$.

Problem: We want to find $\succ$ automatically and we have to check automatically whether

$s \to_{\mathcal{R}} t$ implies $s \succ t$. ← For all (infinitely many) terms s and t.

Solution: Only examine the finitely many rules $\ell \to r$ in the TRS.

Only check whether $\ell \succ r$ holds for every rule $\ell \to r \in \mathcal{R}$.

Is this sufficient for termination of $\mathcal{R}$?

No: $\mathcal{R}$: $\text{endless}(x) \to \text{endless}(s(x))$

$\succ$: let $s \succ t$ hold iff $s = \text{endless}(x)$ and

$t = endless(s(x))$

Clearly, $\succ$ is well founded and we have $\ell \succ r$ for the rule $\ell \to r \in \mathcal{R}$. But $\mathcal{R}$ is not terminating.

<u>Reason</u>: $\succ$ is not stable, but $\to_{\mathcal{R}}$ is stable:

$$endless(x) \to_{\mathcal{R}} endless(s(x)) \to_{\mathcal{R}} endless(s^2(x)) \to_{\mathcal{R}} \dots$$

<u>Solution</u>: Given a TRS $\mathcal{R}$, find a well-founded stable relation $\succ$ such that $\ell \succ r$ for all rules $\ell \to r \in \mathcal{R}$.

<u>Is this correct?</u>

NO: $\mathcal{R}$: $infty(x) \to s(infty(x))$

$\succ$: $t_1 \succ t_2$ iff t_1 starts with "infty" and t_2 starts with "s".

Clearly, $\succ$ is well founded and stable.
Moreover, $\ell \succ r$ holds for $\ell \to r \in \mathcal{R}$. But $\mathcal{R}$ is not terminating:

$$infty(x) \to_{\mathcal{R}} s(infty(x)) \to_{\mathcal{R}} s^2(infty(x)) \to_{\mathcal{R}} \dots$$

<u>Reason</u>: $\to_{\mathcal{R}}$ is monotonic, but $\succ$ is not monotonic.

<u>Solution</u>: To prove termination of a TRS $\mathcal{R}$, we have to find a well-founded, stable, monotonic relation $\succ$

such that $l > r$ for all $l \to r \in R$. (This is indeed correct.)

How can one construct such relations?

- subterm relation $\rhd$ ($s \rhd t$ iff t is a proper subterm of s)

  is well founded, stable, ($s \rhd t$ implies $s\sigma \rhd t\sigma$)

  but not monotonic:

  $f(g(x)) \rhd g(x) \Rightarrow f(g(h(y))) \rhd g(h(\underline{y}))$

  $f(x) \rhd x$, but

  $g(f(x)) \not\rhd g(x)$

- Comparing terms by their size: $\to_{||}$

  $s \to_{||} t$ iff $|s| > |t|$, where $|s|$ is the <u>size</u> of s, i.e., the number of fct. symbols and variables in s.

  $\to_{||}$ is well founded, monotonic, $\left(\begin{array}{l} |s| > |t| \Rightarrow \\ |f(s)| > |f(t)| \end{array} \right)$ etc.

  but not stable:

  $f(x) \to_{||} y$, $\quad \sigma = \{y / f(f(x))\}$

  but $f(x)\sigma \not\to_{||} y\sigma$

To obtain a simple and intuitive relation $>$ that is well founded, stable, and monotonic:

Develop a variant of $\rhd$ which is also monotonic.

Def 4.3.1 (Embedding Order)

The *embedding order* is a relation on $\mathcal{T}(\Sigma, \mathcal{V})$ where

$s \succ_{emb} t$ iff

- $s = f(s_1, \dots, s_n)$ and $s_i \succeq_{emb} t$ for some $i \in \{1, \dots, n\}$ or
- $s = f(s_1, \dots, s_n)$, $t = f(t_1, \dots, t_n)$, $s_i \succ_{emb} t_i$ for some $i \in \{1, \dots, n\}$, and $s_j \succeq_{emb} t_j$ for all $j \in \{1, \dots, n\}$ with $j \neq i$.

Here, $\succeq_{emb}$ is the reflexive closure of $\succ_{emb}$, i.e.,

$s \succeq_{emb} t$ iff $s \succ_{emb} t$ or $s = t$.

For example:

$s(\underline{plus}(\underline{infty}(s(\underline{x})), \underline{y})) \succ_{emb} plus(infty(x), y)$, since

$plus(infty(s(x)), y) \succ_{emb} plus(infty(x), y)$, since

$infty(s(x)) \succ_{emb} infty(x)$ and $y \succeq_{emb} y$, since

$s(x) \succ_{emb} x$, since

$x \succeq_{emb} x$.

Checking whether $s \succ_{emb} t$ holds can easily be done automatically (Def. 4.3.1 corresponds to a recursive algorithm).

To use $>_{emb}$ for termination proofs, we have to ensure that it has the desired properties:

<u>Lemma 4.3.2</u> (Properties of $>_{emb}$)

The embedding order is well founded, stable, monotonic, and transitive.

<u>Proof</u>: <u>Well-Foundedness</u>

- $>_{emb}$ contains the subterm relation $\left(\begin{array}{l} s \rhd t \Rightarrow \\ s >_{emb} t \end{array} \right)$
- $\rightarrow_{||}$ contains $>_{emb}$ $\left(\begin{array}{l} s >_{emb} t \Rightarrow \\ s \rightarrow_{||} t \end{array} \right)$

This can be proved by structural induction on s.

Thus, the size decreases with each $>_{emb}$-step $\Rightarrow$ $>_{emb}$ is well founded.

<u>Stability</u>

Prove $s >_{emb} t \Rightarrow s\sigma >_{emb} t\sigma$ by structural induction on s.

<u>Case 1</u>: $s = f(s_1, \dots, s_n)$, $s_i \geq_{emb} t$

By the ind. hyp: $s_i\sigma \geq_{emb} t\sigma$

Therefore: $s\sigma = f(s_1\sigma, \dots, s_i\sigma, \dots, s_n\sigma) >_{emb} t\sigma$.

<u>Case 2</u>: $s = f(s_1, \dots, s_n)$, $t = f(t_1, \dots, t_n)$, $s_i >_{emb} t_i$, $s_j \geq_{emb} t_j$ for all $j \neq i$.

By the ind. hyp: $s_i\sigma >_{emb} t_i\sigma$,

$s_j\sigma \succeq_{emb} t_j\sigma$ for all $j \neq i$.

Therefore: $s\sigma = f(s_1\sigma, \dots, s_n\sigma) \succ_{emb} t\sigma = f(t_1\sigma, \dots, t_n\sigma)$.

Monotonicity and Transitivity are proved analogously. $\boxtimes$

<u>Def. 4.33.</u> (Reduction relation + order)

A relation on terms which is well founded, stable, and monotonic is called a <u>reduction relation</u>.

A transitive reduction relation is called a <u>reduction order</u>.

In general, an <u>order</u> is a transitive and <u>antisymmetric</u> relation.

$X \succ Y$ and $Y \succ X$ implies $X = Y$

Every well-founded relation is <u>asymmetric</u>, and therefore also antisymmetric.

$X \succ Y$ implies $Y \not\succ X$

<u>Thm 4.34.</u> (Termination Proofs with Reduction Relations, (Manna + Ness, 1970))

A TRS $\mathcal{R}$ terminates iff there exists a reduction relation $\succ$ with $l \succ r$ for all

$\ell \to r \in \mathcal{R}$.

↖ This Thm allows automated termination proofs. But termination remains undecidable, since one doesn't know which reduction relation $\succ$ to choose for a TRS $\mathcal{R}$.

Proof of Thm 4.3.4.:

"$\Leftarrow$": Let $\succ$ be a reduction relation with $\ell \succ r$ for all $\ell \to r \in \mathcal{R}$.

We show that $s \to_{\mathcal{R}} t$ implies $s \succ t$.

(This implies termination of $\mathcal{R}$, since $\succ$ is well founded)

$s \to_{\mathcal{R}} t$ means that there is a position π and a substitution σ such that

$$s|_\pi = \ell\sigma \quad \text{and} \quad t = s[r\sigma]_\pi$$

for some $\ell \to r \in \mathcal{R}$.

Thus: $\ell \succ r \underset{\text{as } \succ \text{ is stable}}{\Rightarrow} \ell\sigma \succ r\sigma \underset{\text{as } \succ \text{ is monotonic}}{\Rightarrow} \overbrace{s[\ell\sigma]_\pi}^{s} \succ \overbrace{s[r\sigma]_\pi}^{t}$

"$\Rightarrow$": Let $\mathcal{R}$ be terminating.

We choose $\succ$ to be $\to_{\mathcal{R}}^{+}$:

- $\ell \to_{\mathcal{R}} r$ holds for all $\ell \to r \in \mathcal{R}$
- $\to_{\mathcal{R}}$ is well founded, since $\mathcal{R}$ terminates
- $\to_{\mathcal{R}}$ is stable + monotonic (Lemma 3.1.13)

$\boxtimes$

Goal: Find suitable reduction relations $\succ$ where "$s \succ t$" can be checked automatically. If $\ell \succ r$ holds for all rules $\ell \to r$, then we have proved termination. Otherwise, we have no information.

Ex 4.3.5: Termination of a minus-TRS: works with $\succ_{emb}$

Termination of plus:

$$\mathrm{plus}(\mathcal{O}, Y) \to Y$$

$$\mathrm{plus}(\mathrm{succ}(X), Y) \to \mathrm{succ}(\mathrm{plus}(X, Y))$$

To prove termination with $\succ_{emb}$, we have to show:

$$\mathrm{plus}(\mathcal{O}, Y) \succ_{emb} Y$$

$$\mathrm{plus}(\mathrm{succ}(X), Y) \succ_{emb} \mathrm{succ}(\mathrm{plus}(X, Y))$$

This does not hold!

Embedding order is too weak

(in particular, when comparing terms with different root symbols).